Local amplitude equation from non-local dynamics

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Abstract. – We derive a closed equation for the shape of the free surface of a magnetic fluid subject to an external magnetic field. The equation is strongly non-local due to the long range character of the magnetic interaction. We develop a systematic multiple scale perturbation expansion in which the non-locality is reduced to the occurrence of the Hilbert transform of the surface profile. The resulting third order amplitude equation describing the slow modulation of the basic pattern is shown to be purely local.

Introduction. – The emergence of spatio-temporal order in distributed systems can often be theoretically analyzed in terms of amplitude equations [1, 2]. Generically these amplitude equations are non-linear partial differential equations for the slow time and space variations of an envelope function of unstable modes. As such they reflect the *local* character of the underlying dynamics.

In several interesting cases, however, the dynamics of the system is substantially influenced by *non-local* interactions. These may arise, e.g., from a mean flow in convection problems [3, 4, 5, 6], from electric and magnetic fields in solids [7, 8] or electrochemical systems [9], from long range elastic interactions [10], or from global couplings in systems with chemical reactions [11, 12, 13].

Since the type and stability of the emerging patterns is usually modified if non-local interactions are present the corresponding amplitude equations are expected to exhibit some degree of non-locality as well. From a phenomenological point of view one may therefore be tempted to simply add to the standard form of an amplitude equation non-local terms complying with the relevant symmetries of the system [14, 15]. A systematic derivation of the amplitude equation from the basic non-local dynamics has been accomplished in a few cases only. [7, 11, 6, 10, 13].

In the present letter we investigate the formation of static patterns on the free surface of a magnetic fluid subject to an external magnetic field. In this process the non-local character of the magnetic interaction is of vital importance. We establish a closed, strongly non-local equation for the free surface of the magnetic fluid and systematically derive an amplitude equation for the surface deflection in the vicinity of the critical field strength. Although the

corresponding linear operator is non-local the lowest order amplitude equation is found to be *local* and to reproduce previous results from a perturbative analysis of the free energy [18,19].

Magnetic fluids are suspensions of ferromagnetic nanoparticles in suitable carrier liquids behaving as super-paramagnetic Newtonian liquids [16]. If the strength of a magnetic field perpendicular to the flat free surface of a magnetic fluid exceeds a threshold value the surface is known to become unstable due to the normal field or Rosensweig instability [17] and a hexagonal or square pattern of fluid peaks develops [17,18].

In order to keep the analysis simple it is convenient to consider a situation with just one unstable mode analogous to the roll solution in hydrodynamic systems. As was known experimentally for some time [20] and has been clarified theoretically recently [21] the corresponding "rigde" pattern on a magnetic fluid surface may be induced by an *oblique* magnetic field. In this case the field component tangential to the flat surface suppresses surface deflections in this direction.

Basic equations. – We consider an incompressible magnetic fluid of infinite depth, density ρ and constant permeability $\mu = \mu_0\mu_r$ in a magnetic field $\mathbf{H}_0$ which, in the absence of any magnetically permeable material, is homogeneous and of the form $\mathbf{H}_0 = H_\mathrm{Z}\mathbf{e}_z + H_\mathrm{X}\mathbf{e}_x$. The free surface between the fluid and the magnetically impermeable air above is described by $z = \zeta(x,y)$ (see fig. 1). The surface tension is denoted by σ and gravity acts parallel to the z-axis, $\mathbf{g} = -g\mathbf{e}_z$. We assume the horizontal magnetic field component $H_\mathrm{X}\mathbf{e}_x$ to be strong enough to suppress surface deflections in x-direction rendering the problem effectively two dimensional.

The static surface profile $\zeta(y)$ is then determined by the pressure equilibrium at the surface [22,16]

$$\rho g\zeta - \sigma\frac{\partial_y^2\zeta}{[1+(\partial_y\zeta)^2]^{3/2}} - \frac{\mu_0(\mu_\mathrm{r}-1)}{2}\mathbf{H}^2\Big|_\zeta - \frac{\mu_0(\mu_\mathrm{r}-1)^2}{2}H_n^2\Big|_\zeta = p. \tag{1}$$

Here $\mathbf{H}\big|_\zeta$ and $H_n\big|_\zeta$ denote the magnetic field and its normal component, respectively, *in* the fluid at $z=\zeta(y)$. The air pressure p on the right hand-side is assumed to be constant. The first two terms of eq. (1) are the standard expressions describing the influence of gravity and surface tension. The remaining terms characterize the impact of the magnetic field on the surface profile $\zeta(y)$.

The magnetic field $\mathbf{H}(y,z)$ has to fulfill the magneto-static Maxwell equations

$$\nabla\cdot\mathbf{B} = 0 \quad \text{and} \quad \nabla\times\mathbf{H} = \mathbf{0}, \tag{2}$$

where the magnetic induction $\mathbf{B}(y,z) = \mu_0\mu_\mathrm{r}\mathbf{H}(y,z)$ is a linear function of the magnetic field. The field equations are supplemented by the boundary conditions

$$(\mathbf{B}^{(a)} - \mathbf{B}^{(f)})\big|_\zeta\cdot\mathbf{n} \;= 0 \tag{3}$$

$$(\mathbf{H}^{(a)} - \mathbf{H}^{(f)})\big|_\zeta\times\mathbf{n} \;= \mathbf{0}, \tag{4}$$

which describe the feedback of the surface profile on the magnetic field. Here

$$\mathbf{n} = \frac{(0,-\partial_y\zeta,1)}{\sqrt{1+(\partial_y\zeta)^2}} \tag{5}$$

is the normal vector on the free surface pointing outwards and the upper indices $^{(a)}$ and $^{(f)}$ refer to the air above and the fluid below the interface, respectively. The conditions

$$\mathbf{H}(y,z)\to\begin{cases} H_\mathrm{Z}\,\mathbf{e}_z & \text{if} \quad z\to\infty \\ H_\mathrm{Z}/\mu_r\,\mathbf{e}_z & \text{if} \quad z\to-\infty \end{cases} \tag{6}$$

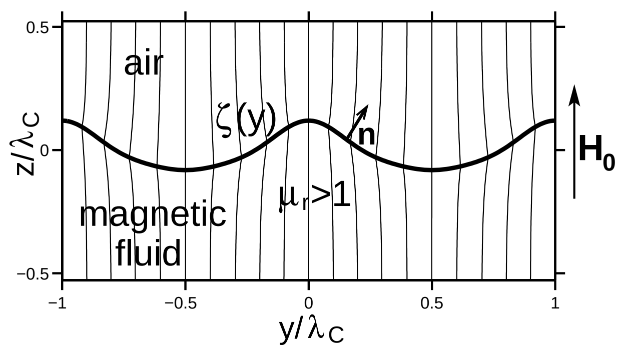

Fig. 1

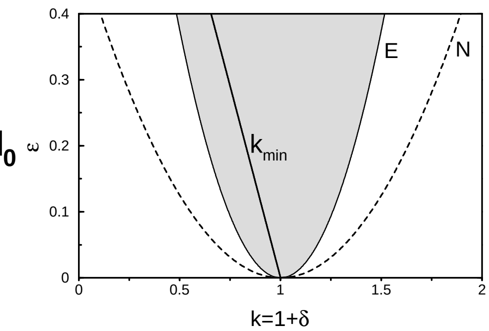

Fig. 2

Fig. 1 – Static surface profile $\zeta(y)$ and magnetic field lines for a magnetic fluid with $\mu_\mathrm{r} = 2$ in an external magnetic field corresponding to $\epsilon = 0.0725$. The critical wavelength $\lambda_\mathrm{c} = 2\pi/k_\mathrm{c}$ is typically of the order of 1 cm.

Fig. 2 – Stability diagram for a magnetic fluid with $\mu_\mathrm{r} = 2$ in an overcritical magnetic field. ϵ is related to the magnetic field strength by (8), δ denotes the deviation from the critical wavenumber $k_\mathrm{c} = 1$. N is the neutral curve of the linear instability, E denotes the Eckhaus boundary following from the amplitude equation (29). Also shown is the wavenumber k_min minimizing the free energy as determined in [19].

ensure that the magnetic field asymptotically approaches its externally prescribed values.

It is convenient to rescale lenght by the capillary length $k_\mathrm{c}^{-1} = \sqrt{\sigma/\rho g}$, pressure by $\sqrt{\rho g \sigma}$ and magnetic field by the critical field strength of the Rosensweig instability

$$H_\mathrm{c} = \sqrt{\frac{2\mu_\mathrm{r}(\mu_\mathrm{r}+1)\sqrt{\rho g \sigma}}{\mu_0(\mu_\mathrm{r}-1)^2}}. \tag{7}$$

Moreover we introduce the supercriticality parameter

$$\epsilon = H_\mathrm{Z}^2 - 1 \tag{8}$$

and the parameter

$$\eta = \frac{\mu_\mathrm{r}-1}{\mu_\mathrm{r}+1} \tag{9}$$

characterizing the magnetic susceptibility of the fluid. Denoting the tangential component of the magnetic field by H_t eq.(1) then assumes the dimensionless form

$$\zeta - \frac{\partial_y^2 \zeta}{[1+(\partial_y \zeta)^2]^{3/2}} - \frac{\mu_\mathrm{r}}{\eta} H_t^2\Big|_\zeta - \frac{\mu_\mathrm{r}^2}{\eta} H_n^2\Big|_\zeta = -\frac{1+\epsilon}{\eta}, \tag{10}$$

where the value of p was determined from the reference state $\zeta \equiv 0$.

The dependance of the magnetic field on the surface deflection via the boundary conditions (3) and (4) is rather implicit. To make the non-locality of the problem more explicit we take advantage to the *linearity* of the magnetic field problem and employ a formal Greens function solution for the magnetic field. To this end we write the field in the form

$$\mathbf{H}(y,z) = \frac{\mu_\mathrm{r}-1}{2\mu_\mathrm{r}} H_\mathrm{Z} \int_{-\infty}^{+\infty} \frac{dy'}{2\pi}\, q(y')\, \frac{(y-y')\mathbf{e}_y + (z-\zeta(y'))\mathbf{e}_z}{(y-y')^2 + (z-\zeta(y'))^2} + \frac{1+\mu_\mathrm{r}}{2\mu_\mathrm{r}} H_\mathrm{Z}\, \mathbf{e}_z, \tag{11}$$

where the so far unknown magnetic source density $q(y)$ is nonzero at the surface $z = \zeta(y)$ only. The field (11) already fulfills the Maxwell equations (2) as well as the boundary condition (4). The remaining boundary condition (3) gives rise to an equation for $q(y)$. To derive this equation we observe that the normal component of the field (11) is discontinuous at the interface,

$$H_n^{(\mathrm{a})} - H_n^{(\mathrm{f})} = \frac{\mu_\mathrm{r} - 1}{2\mu_\mathrm{r}} H_\mathrm{Z} \frac{q(y)}{\sqrt{1 + (\partial_y \zeta)^2}}. \tag{12}$$

Moreover, interpreting the integral in (11) for $z = \zeta(y)$ as Cauchy principal value we find for the field directly *at* the surface $H_n(y, \zeta(y)) = (H_n^{(\mathrm{a})} + H_n^{(\mathrm{f})})/2$. Together with (3) this gives rise to

$$\frac{2\mu_\mathrm{r}}{\mu_\mathrm{r} + 1}\, H_n(y, \zeta(y)) = H_\mathrm{Z} \frac{q(y)}{2\sqrt{1 + (\partial_y \zeta(y))^2}}, \tag{13}$$

and using this relation in eq. (11) we obtain for $q(y)$ the linear integral equation

$$q = 1 - \eta (\partial_y \zeta) \mathcal{S} q + \eta \mathcal{T} q. \tag{14}$$

The operators $\mathcal{S}$ and $\mathcal{T}$ act on bounded functions $f(y)$ and are defined by

$$(\mathcal{S} f)(y) = \fint_{-\infty}^{+\infty} \frac{dy'}{\pi} \frac{y - y'}{(y - y')^2 + (\zeta(y) - \zeta(y'))^2}\, f(y') \tag{15}$$

and

$$(\mathcal{T} f)(y) = \fint_{-\infty}^{+\infty} \frac{dy'}{\pi} \frac{\zeta(y) - \zeta(y')}{(y - y')^2 + (\zeta(y) - \zeta(y'))^2}\, f(y'). \tag{16}$$

where $\fint$ denotes the Cauchy principal value of the integral. Note that due to the constant term on the rhs of (14) also the asymptotic conditions (6) are fulfilled.

With the help of (11) we can express the magnetic field contributions in (10) in terms of the source density $q(y)$ and obtain the pressure balance in the form

$$0 = \zeta - \frac{\partial_y^2 \zeta}{[1 + (\partial_y \zeta)^2]^{3/2}} + \frac{1 + \epsilon}{\eta} - \frac{1 + \epsilon}{1 + (\partial_y \zeta)^2} \left[\frac{\big(\eta [1 + (\partial_y \zeta)^2] \mathcal{S} q + (\partial_y \zeta) q\big)^2}{(1 + \eta)\eta(1 - \eta)} + \frac{q^2}{\eta} \right] \tag{17}$$

Eqs.(17) and (14) are a closed system of integro-differential equations for the stationary surface profile $\zeta(y)$ and the related source density $q(y)$.

While the kernels of the operators $\mathcal{S}$ and $\mathcal{T}$ are nonlinear functions of the surface deflection $\zeta(y)$ eq. (14) is a *linear* equation for the source density $q(y)$. For given $\zeta(y)$ it can hence be solved formally using the Neumann series

$$q = 1 + \sum_{n=1}^{\infty} \eta^n\, [\mathcal{T} - (\partial_y \zeta) \mathcal{S}]^n\, 1. \tag{18}$$

Replacing $q(y)$ in eq. (17) by this series we obtain a closed nonlinear integro-differential equation for the free surface profile $\zeta(y)$. This equation contains products of an infinite number of integral operators and is per se of limited use only. Nevertheless it forms a suitable starting point for a systematic multiple scale perturbation analysis of the weakly non-linear regime. On the other hand this expansion can also be performed directly on the system (17) and (14) which is what we do in the following.

Weakly nonlinear analysis. – We consider slightly supercritical fields $0 \leq \epsilon \ll 1$ and assume that both the surface deflection $\zeta(y)$ and the source density $q(y)$ can be expanded in powers of $\epsilon^{1/2}$:

$$\zeta = \epsilon^{1/2}\,\zeta_1 + \epsilon\,\zeta_2 + \epsilon^{3/2}\,\zeta_3 + \ldots \tag{19}$$

$$q = 1 + \epsilon^{1/2}\,q_1 + \epsilon\,q_2 + \epsilon^{3/2}\,q_3 + \ldots. \tag{20}$$

For the resulting expansions of the linear integral operators $\mathcal{S}$ and $\mathcal{T}$ it is convenient to start with the series representations

$$(\mathcal{S}f)(y) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \int_{-\infty}^{+\infty} \frac{dy'}{\pi}\, \frac{\partial_{y'}^{2n}\left[(\zeta(y)-\zeta(y'))^{2n} f(y')\right]}{(y'-y)} \tag{21}$$

and

$$(\mathcal{T}f)(y) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} \int_{-\infty}^{+\infty} \frac{dy'}{\pi}\, \frac{\partial_{y'}^{2n+1}\left[(\zeta(y)-\zeta(y'))^{2n+1} f(y')\right]}{(y'-y)}, \tag{22}$$

arising from an expansion of the denominators in (15) and (16) and appropriate partial integration. The central building block in these expansions is the *Hilbert transform* defined for bounded functions $h(y)$ by

$$(\mathcal{H}h)(y) = \int_{-\infty}^{+\infty} \frac{dy'}{\pi}\, \frac{h(y')}{(y'-y)}. \tag{23}$$

Using (19) in eqs.(21) and (22) gives rise to expansions of $\mathcal{S}$ and $\mathcal{T}$ in powers of $\epsilon^{1/2}$ which together with (19) and (20) are used in (17) and (14). Matching powers in ϵ then yields a hierarchy of linear equations for the expansion coefficients of ζ and q in the usual way.

To order $O(\epsilon^{1/2})$ we find

$$\mathcal{L}\,\zeta_1 = \zeta_1 + 2\partial_y \mathcal{H}\zeta_1 - \partial_y^2\,\zeta_1 = 0. \tag{24}$$

The linear operator $\mathcal{L}$ is *non-local* due to the occurence of the Hilbert transform originating from the non-local magnetic interaction. With the help of $\mathcal{H}\exp(iky) = \mathrm{sgn}(k)\,i\exp(iky)$ the solution of this equation is easily found to be

$$\zeta_1(y) = Ae^{iy} + A^* e^{-iy} = Ae^{iy} + \text{c.c.}\,, \tag{25}$$

with a complex valued amplitude A. As it should the first order of the perturbation expansion hence reproduces the results of the linear stability analysis, in particular $k_c = 1$. Note also that under the standard scalar product the operator $\mathcal{L}$ is *self-adjoint*, $\mathcal{L}^\dagger = \mathcal{L}$.

For $\epsilon > 0$ not only the critical mode but a whole narrow *band* of modes with k-values around k_c are unstable. As is well known this gives rise to long wavelength modulations of the basic pattern [1]. To take account of this possibility we allow for a slow spatial dependence of the amplitude, $A(Y)$, formalized by the substitution $\partial_y \to (\partial_y + \epsilon^{1/2}\partial_Y)$. Using the properties of the Fourier transform of $A(Y)$ one can also show $\mathcal{H}A(Y)\exp(iky) = A(Y)\mathcal{H}\exp(iky)$.

To order $O(\epsilon)$ we then find

$$\mathcal{L}\zeta_2 = 2\eta A^2 e^{2iy} + \text{c.c.}\,, \tag{26}$$

with the solution

$$\zeta_2(y) = 2\eta A^2 e^{2iy} + \text{c.c.}\,. \tag{27}$$

Here the amplitude of the solution of the homogeneous equation was set equal to zero in accordance with the general requirement that all ζ_n with $n > 1$ should be orthogonal to ζ_1.

Eventually, to order $O(\epsilon^{3/2})$ we get the inhomogeneous equation

$$\mathcal{L}\zeta_3 = 2A\,e^{iy} + \partial_Y^2 A\,e^{iy} - \left[\frac{5}{2} - 8\eta^2\right]|A|^2 A\,e^{iy} + 4i\eta A\,\partial_Y A\,e^{2iy} + \left[\frac{1}{2} + 16\eta^2\right] A^3\,e^{3iy} + \text{c.c.}\,. \tag{28}$$

Since $\mathcal{L}$ is singular (cf. eq. (24)) the right hand side of this equation must be orthogonal to the zero eigenfunctions of $\mathcal{L}^\dagger$ which are $e^{\pm iy}$. Returning to unscaled variables we hence find the amplitude equation

$$0 = \epsilon A + \frac{1}{2}\,\partial_y^2 A - \left[\frac{5}{4} - 4\eta^2\right]|A|^2 A. \tag{29}$$

Despite the non-local character of the linear operator $\mathcal{L}$ this equation is *local*. The terms without derivatives are identical with those obtained from a perturbative analysis of the free energy of the static surface deflection [18, 19]. In particular we reproduce the result that for $\eta > \sqrt{5}/4$ or equivalently $\mu_r > 3.5353$ the cubic term becomes positive and higher order non-linear terms are needed to saturate the instability [23, 24].

After imposing the solvability condition (29) we can integrate (28) to get

$$\zeta_3(y) = 4i\eta A\,\partial_Y A\,e^{2iy} + \left[\frac{1}{8} + 4\eta^2\right] A^3\,e^{3iy} + \text{c.c.}\,. \tag{30}$$

Eqs. (25), (27), and (30) with $\partial_Y A = 0$ have been used to make the plot of $\zeta(y)$ shown in Fig. 1.

Building on the dispersion relation for surface waves on ferrofluids [25, 26] also a slow *time* dependence of the amplitude A with the corresponding term on the left hand side of (29) can be included. This then allows e.g. to investigate the stability of solutions with slow spatial modulations of the form

$$A(y) = \tilde{A}\,e^{i\delta y}. \tag{31}$$

A straightforward analysis [1] yields as boundary for the associated Eckhaus instability

$$\delta_{\mathrm{E}}^2(\epsilon) = 2\epsilon/3, \tag{32}$$

which is shown in Fig. 2.

Conclusion. – To determine the equilibrium shape of the free surface of a magnetic fluid in an external magnetic field is a highly non-local problem. Eliminating the magnetic field problem by introducing a suitable magnetic charge density on the free surface we have derived a closed set of two non-local equations for the determination of the surface profile. In a systematic multiple scale perturbation expansion of these equations the non-locality results in a non-local linear operator $\mathcal{L}$ involving the Hilbert transform. On the other hand, the third order amplitude equation for the surface deflection was shown to be local. An extension of the perturbation expansion to other patterns as hexagons and squares or higher orders is straightforward though tedious.

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